

Localization of bulk matter fields on a pure de Sitter thick braneworld

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ABSTRACT: In this paper we investigate the localization and mass spectra of various matter fields with spin 0, 1 and 1/2 on a thick brane generated by pure 4D and 5D positive cosmological constants without bulk scalar fields. For spin 0 scalar and spin 1 vector fields, the potentials of the Kaluza–Klein (KK) modes in the corresponding Schrödinger equations are modified Pöschl–Teller potentials, which lead to the localization of the scalar and vector zero modes on the brane as well as to mass gaps in the mass spectra. For spin 1/2 fermions, we introduce the bulk mass term $MF(z)\bar{\Psi}\Psi$ in the action and four different cases are investigated. Localization of the massless left–chiral fermion zero mode is feasible for just two cases of $F(z)$. In the first one we obtain Schrödinger equations with modified Pöschl–Teller potentials with the corresponding mass gaps for both left– and right–chiral fermions, the number of massive KK bound states is *finite* (determined by the ratio M/b) and is the same for left– and right–chiral modes. In the second case we get a family of solutions with an *infinite* number of bound states where the mass spectra for left– and right–chiral KK fermion modes are discrete. A special case resembles the one–dimensional quantum harmonic oscillator problem. For both of these cases, the mass spectrum of right–chiral fermions is shifted with respect to the mass spectrum of the left–chiral ones.

KEYWORDS: Large Extra Dimensions, Field Theories in Higher Dimensions.

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Contents

1. Introduction	1
2. Review of the thick braneworld generated by pure curvature	2
3. Localization of various matter fields on a pure de Sitter thick brane	3
3.1 Spin-0 scalar fields	4
3.2 Spin-1 vector fields	6
3.3 Spin-1/2 fermion fields	9
3.3.1 Case I: $F(z) = \varepsilon(z)$	11
3.3.2 Case II: $F(z) = \tanh(Hz)$	11
3.3.3 Case III: $F(z) = \sinh(Hz)$	12
3.3.4 Case IV: $F(z) = (Hz)^{2k+1}e^{-A(z)}$	15
4. Conclusion and discussion	20
5. Acknowledgement	21

1. Introduction

The idea that our observed four-dimensional universe can be considered as a brane, which is embedded in a higher dimensional space-time, has received considerable attention. In the string/M-theory context, branes naturally appear and provide a novel mode for discussing phenomenological and cosmological issues related to extra dimensions. The proposal that extra dimensions may not be compact [1, 2, 3, 4, 5, 6] or large [7, 8] can supply new insights for solving the gauge hierarchy problem [8] and the cosmological constant problem [1, 3, 9]. The framework of brane scenarios is that gravity is free to propagate in all dimensions, however all matter fields (electromagnetic, Yang-Mills, etc.) are confined to a 3-brane with no contradiction with present gravitational experiments [1, 2, 8]. In Refs. [4, 5], an alternative scenario, the so-called Randall-Sundrum (RS) braneworld model showed that the effective four-dimensional gravity could be recovered even in the case of non-compact extra dimensions.

A more natural theory arises in the framework of thick brane scenarios, which are generally based on gravity coupled to one or several scalars [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56]. In these scenarios, the scalar fields do not play the role of bulk fields, but provide the “material” from which the thick branes are made of. For some comprehensive reviews about thick branes please see Refs. [57, 58, 59, 60, 61, 62].

In brane world scenarios, there is an interesting issue: whether various bulk fields could be confined to the brane by a natural mechanism. Generally, massless scalar fields [21] and gravitons [5] can be localized on branes of different types. Vector fields can be localized on the RS brane in some higher-dimensional cases [22] or on thick dS branes and Weyl thick branes [23]. The feature of fermion localization is very important. In general, if one does not introduce the coupling between the fermions and the bulk scalars, the fermions do not have normalizable zero modes in five and six dimensions [21, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38]. In some cases, with scalar-fermion coupling, a single bound state and a continuous gapless spectrum of massive fermion KK states can be obtained [23, 39]. In some other brane models, one can obtain finite discrete KK states (mass gap) and a continuous gapless spectrum starting at a positive m^2 [41, 42, 43, 44].

Recently, a pure de Sitter thick brane has been investigated in Ref. [55]. This thick brane is not made of bulk scalars but rather is modeled by an intriguing relation between the curvatures generated by the five- and four-dimensional cosmological constants. The localization of gravity and corrections to Newton's law have already been studied in this model. In this paper we will explore the localization and mass spectra of various matter fields with spin 0, 1 and 1/2 on this thick brane scenario. For the spin 0 scalars and the spin 1 vectors, the zero modes can be localized on the brane, and there exists a mass gap in the mass spectra. However, the localization of the spin 1/2 fermions is very special since there is no scalar field to couple with in this model. Thus, in order to trap the fermions we introduce the bulk mass term $MF(z)\bar{\Psi}\Psi$, so that the character of the localization is different for different forms of the function $F(z)$.

The organization of this paper is as follows: In Sec. 2, we first review the so-called pure de Sitter thick brane without the inclusion of bulk scalar fields. Then, in Sec. 3, we investigate the localization and mass spectra of various matter fields with spin 0, 1 and 1/2 on this thick brane. Finally, our conclusion is given in Sec. 4.

2. Review of the thick braneworld generated by pure curvature

We start with the following five-dimensional action of a thick braneworld model, avoiding at all the use of background scalar fields [55]

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} (R - 2\Lambda_5), \quad (2.1)$$

where R and Λ_5 are the five-dimensional scalar curvature and the bulk cosmological constant, and $\kappa_5^2 = 8\pi G_5$ with G_5 the five-dimensional Newton constant. From this action, the Einstein equations with a cosmological constant in five dimensions are easy to get

$$G_{MN} = -\Lambda_5 g_{MN}. \quad (2.2)$$

The most general five-dimensional metric compatible with an induced 3-brane with spatially flat cosmological background can be taken to be

$$\begin{aligned} ds^2 &= g_{MN} dx^M dx^N = e^{2A(y)} [\hat{g}_{\mu\nu}(x) dx^\mu dx^\nu] + dy^2 \\ &= e^{2A(y)} [-dt^2 + a^2(t)(dx_1^2 + dx_2^2 + dx_3^2)] + dy^2, \end{aligned} \quad (2.3)$$

where $e^{2A(y)}$ and $a(t)$ are the warp factor and the scale factor of the brane, and y stands for the extra dimensional coordinate.

By considering this metric (2.3), we can compute the components of the Einstein tensor and the set of Einstein equations can be reduced to a very simple system [55]:

$$A''(y) = \frac{1}{3} \left(\frac{2\dot{a}^2}{a^2} - \frac{5\ddot{a}}{a} \right) e^{-2A(y)}, \quad (2.4a)$$

$$A'^2(y) = \frac{1}{6} \left[\left(\frac{5\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) e^{-2A(y)} - \Lambda_5 \right], \quad (2.4b)$$

where the prime and the dot denote derivative with respect to y and t , respectively. By deriving the Eq. (2.4b) with respect to y and comparing it with Eq. (2.4a), we can obtain a second order differential equation for the scale factor

$$a\ddot{a} - \dot{a}^2 = 0. \quad (2.5)$$

The general solution of Eq. (2.5) is $a(t) = ce^{Ht}$ with c and H arbitrary constants, and we can choose the scale factor corresponding to a de Sitter four-dimensional cosmological background

$$a(t) = e^{Ht}, \quad (2.6)$$

since the constant c can be absorbed into a coordinate redefinition. Here H is the de Sitter parameter and $3H^2 = \Lambda_4$ with Λ_4 being the four-dimensional cosmological constant [61]. Since the form of the scale factor has been determined, furthermore the solution for the warp factor can be found [55]:

$$A(y) = \ln \left[\frac{H}{b} \cos(by) \right], \quad (2.7)$$

where b is the constant which is inversely proportional to the thickness of the 3-brane and is determined by the five-dimensional cosmological constant as follows:

$$b^2 = \frac{\Lambda_5}{6}. \quad (2.8)$$

From the warp factor, we come to the conclusion: a thick de Sitter brane is localized around $y = 0$, and the range of the fifth dimension is $-\left|\frac{\pi}{2b}\right| < y < +\left|\frac{\pi}{2b}\right|$. The localization and stability properties of graviton fluctuations on the brane were studied in [55].

3. Localization of various matter fields on a pure de Sitter thick brane

In this section we shall investigate the localization of various bulk matter fields on a pure de Sitter thick brane. Spin-0 scalars, spin-1 vectors and spin-1/2 fermions will be considered by means of gravitational interaction. Certainly, it has been implicitly assumed that the various bulk matter fields considered below make little contribution to the bulk energy so that the solutions given in the previous section remain valid even in the presence of

bulk matter. Thus, these bulk matter fields make little contribution to geometry of the bulk spacetime. The mass spectra of various matter fields on the pure de Sitter thick brane will also be discussed by presenting and analyzing the potential of the corresponding Schrödinger equation for their KK modes.

In order to get mass-independent potentials of the corresponding Schrödinger equation, we will follow Ref. [5] and change the metric given in (2.3) to the following one

$$\begin{aligned} ds^2 &= e^{2A(z)} [\hat{g}_{\mu\nu}(x) dx^\mu dx^\nu + dz^2] \\ &= e^{2A(z)} [-dt^2 + e^{2Ht} (dx_1^2 + dx_2^2 + dx_3^2) + dz^2] \end{aligned} \quad (3.1)$$

by performing the coordinate transformation

$$dz = e^{-A(y)} dy. \quad (3.2)$$

Then, the following expression for z can be obtained

$$z(y) = \int e^{-A(y)} dy = \frac{2}{H} \operatorname{arctanh} \left[\tan \left(\frac{H}{b} y \right) \right]. \quad (3.3)$$

It is easy to see that $z \rightarrow \pm\infty$ as $y \rightarrow \pm \left| \frac{\pi}{2b} \right|$, so the range of z is $-\infty < z < +\infty$. Due to this transformation, the warp factor A can be rewritten as a function of z :

$$A(z) = \ln \left[\frac{H}{b} \operatorname{sech}(Hz) \right]. \quad (3.4)$$

In the following, the localization of various bulk matter fields will be investigated, and it will be seen that the mass-independent potentials can be obtained conveniently with the aid of the metric (3.1).

3.1 Spin-0 scalar fields

We begin by considering the localization of real scalar fields on the thick brane obtained in the previous section, turning to vectors and fermions in the next subsections. Let us start by considering the action of a massless real scalar field coupled to gravity

$$S_0 = -\frac{1}{2} \int d^5x \sqrt{-g} g^{MN} \partial_M \Phi \partial_N \Phi. \quad (3.5)$$

Using the conformal metric (3.1), the equation of motion derived from (3.5) reads

$$\frac{1}{\sqrt{-\hat{g}}} \partial_\mu \left(\sqrt{-\hat{g}} \hat{g}^{\mu\nu} \partial_\nu \Phi \right) + e^{-3A} \partial_z (e^{3A} \partial_z \Phi) = 0. \quad (3.6)$$

Then, by using the KK decomposition $\Phi(x, z) = \sum_n \phi_n(x) \chi_n(z) e^{-3A/2}$ and demanding that Φ_n satisfies the four-dimensional massive Klein-Gordon equation:

$$\left[\frac{1}{\sqrt{-\hat{g}}} \partial_\mu \left(\sqrt{-\hat{g}} \hat{g}^{\mu\nu} \partial_\nu \right) - m_n^2 \right] \phi_n(x) = 0, \quad (3.7)$$

we can obtain the equation for the scalar KK mode $\chi_n(z)$:

$$[-\partial_z^2 + V_0(z)] \chi_n(z) = m_n^2 \chi_n(z), \quad (3.8)$$

which is a Schrödinger equation with the effective potential given by

$$V_0(z) = \frac{3}{2}A'' + \frac{9}{4}A'^2, \quad (3.9)$$

where m_n is the four-dimensional mass of the KK excitation of the scalar field. It is clear that the potential $V_0(z)$ defined in (3.9) is completely determined by the warp factor and, hence, it is a four-dimensional mass-independent potential.

The full five-dimensional action (3.5) can be reduced to the standard four-dimensional action for a massless and a series of massive scalars

$$S_0 = -\frac{1}{2} \sum_n \int d^4x \sqrt{-\hat{g}} \left(\hat{g}^{\mu\nu} \partial_\mu \phi_n \partial_\nu \phi_n + m_n^2 \Phi_n^2 \right), \quad (3.10)$$

when integrated over the extra dimension, with the requirement that Eq. (3.8) is satisfied and the following orthonormalization conditions are obeyed:

$$\int_{-\infty}^{\infty} \chi_m(z) \chi_n(z) dz = \delta_{mn}. \quad (3.11)$$

For the thick brane solution (3.4), the potential (3.9) adopts the form

$$V_0(z) = \frac{9}{4}H^2 - \frac{15}{4}H^2 \text{sech}^2(Hz). \quad (3.12)$$

This potential has a minimum (negative value) $-\frac{3H^2}{2}$ at $z = 0$ and a maximum (positive value) $\frac{9H^2}{4}$ at $z = \pm\infty$ (see Fig. 1). Then, Eq. (3.8) can turn into the well-known Schrödinger equation with $E_n = m_n^2 - \frac{9H^2}{4}$:

$$\left[-\partial_z^2 - \frac{15H^2}{4} \text{sech}^2(Hz) \right] \chi_n = E_n \chi_n. \quad (3.13)$$

For this equation with a modified Pöschl–Teller potential, the energy spectrum of bound states is found to be [12, 23, 42, 43, 55, 63]

$$E_n = -H^2 \left(\frac{3}{2} - n \right)^2 \quad (3.14)$$

or, in terms of the squared mass:

$$m_n^2 = n(3 - n)H^2, \quad (3.15)$$

where n is an integer and satisfies $0 \leq n < \frac{3}{2}$, so that it is clear that there are two bound states. The first one is the ground state with $m_0^2 = 0$, and can be expressed as

$$\chi_0(z) = \sqrt{\frac{2H}{\pi}} \text{sech}^{\frac{3}{2}}(Hz), \quad (3.16)$$

which is just the massless mode and also shows that there is no tachyonic scalar mode. The second one is the first excited state, which corresponds to $n = 1$ and $m_1^2 = 2H^2$, and can be written as

$$\chi_1(z) = \sqrt{\frac{2H}{\pi}} \text{sech}^{\frac{3}{2}}(Hz) \sinh(Hz). \quad (3.17)$$

The shapes of the bound KK modes and the mass spectrum are shown in Fig. 2 and Fig. 3. The continuous spectrum starts at $m^2 = 9H^2/4$.

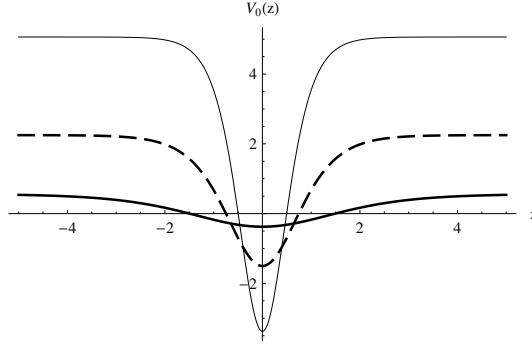


Figure 1: The shape of the potential of the scalar KK modes $V_0(z)$ for different values of H , which is set to $H = 0.5$ for the thick line, $H = 1.0$ for the dashed line, and $H = 1.5$ for the thin line.

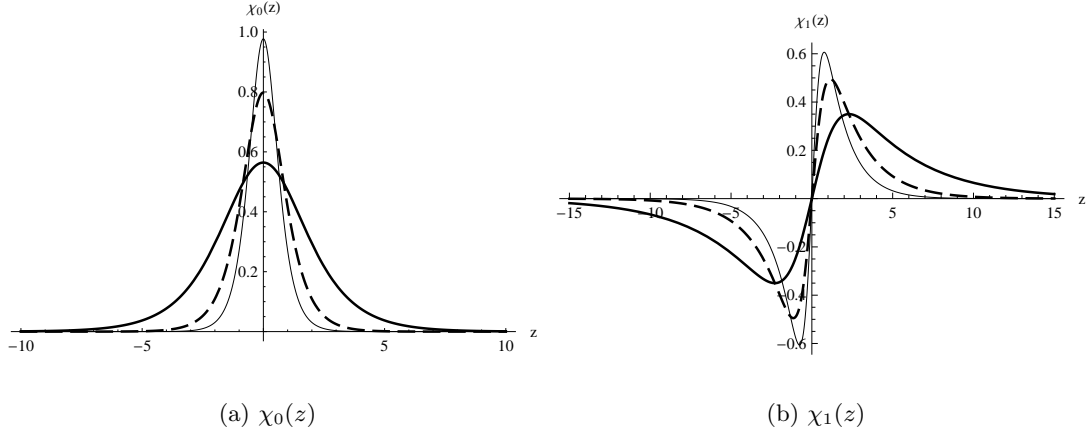


Figure 2: The shape of the scalar zero mode $\chi_0(z)$ in (a), and the first excited state $\chi_1(z)$ in (b). The parameter H is set to $H = 0.5$ for the thick line, $H = 1.0$ for the dashed line, and $H = 1.5$ for the thin line.

3.2 Spin-1 vector fields

We now turn to spin 1 vector fields. We begin with the five-dimensional action of a vector field

$$S_1 = -\frac{1}{4} \int d^5x \sqrt{-g} g^{MN} g^{RS} F_{MR} F_{NS}, \quad (3.18)$$

where $F_{MN} = \partial_M A_N - \partial_N A_M$ is the field strength tensor as usual. From this action the equations of motion are read as follows

$$\frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} g^{MN} g^{RS} F_{NS}) = 0. \quad (3.19)$$

By using the background metric (3.1), the equations of motion can be written as

$$\frac{1}{\sqrt{-\hat{g}}} \partial_\nu \left(\sqrt{-\hat{g}} \hat{g}^{\nu\rho} \hat{g}^{\mu\lambda} F_{\rho\lambda} \right) + \hat{g}^{\mu\lambda} e^{-A} \partial_z (e^A F_{4\lambda}) = 0, \quad (3.20)$$

$$\partial_\mu \left(\sqrt{-\hat{g}} \hat{g}^{\mu\nu} F_{\nu 4} \right) = 0. \quad (3.21)$$

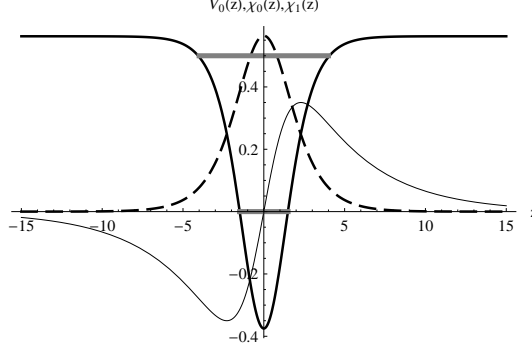


Figure 3: The shape of the potential of scalar KK modes V_0 (thick line), the bound scalar KK modes χ_n (dashed line for χ_0 and thin line for χ_1) and the mass spectrum (thick grey lines) with the parameter $H = 0.5$.

Because the fourth component A_4 has no zero mode in the effective four-dimensional theory, we assume that it is Z_2 -odd with respect to the extra dimension z . Furthermore, in order to be consistent with the gauge invariant equation $\oint dz A_4 = 0$, we choose $A_4 = 0$ by using the gauge freedom. Then, the action (3.18) can be reduced to

$$S_1 = -\frac{1}{4} \int d^5x \sqrt{-g} \left\{ g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + 2e^{-A} g^{\mu\nu} \partial_z A_\mu \partial_z A_\nu \right\}. \quad (3.22)$$

With the decomposition of the vector field $A_\mu(x, z) = \sum_n a_\mu^{(n)}(x) \rho_n(z) e^{-A/2}$ and the orthonormalization conditions

$$\int_{-\infty}^{\infty} \rho_m(z) \rho_n(z) dz = \delta_{mn}, \quad (3.23)$$

the action (3.22) is read

$$S_1 = \sum_n \int d^4x \sqrt{-\hat{g}} \left(-\frac{1}{4} \hat{g}^{\mu\alpha} \hat{g}^{\nu\beta} f_{\mu\nu}^{(n)} f_{\alpha\beta}^{(n)} - \frac{1}{2} m_n^2 \hat{g}^{\mu\nu} a_\mu^{(n)} a_\nu^{(n)} \right), \quad (3.24)$$

where $f_{\mu\nu}^{(n)} = \partial_\mu a_\nu^{(n)} - \partial_\nu a_\mu^{(n)}$ is the four-dimensional field strength tensor. In the above process, it has been required that the vector KK modes $\rho_n(z)$ should satisfy the following Schrödinger equation:

$$[-\partial_z^2 + V_1(z)] \rho_n(z) = m_n^2 \rho_n(z), \quad (3.25)$$

where the mass-independent potential $V_1(z)$ is given by

$$V_1(z) = \frac{H^2}{4} - \frac{3H^2}{4} \text{sech}^2(Hz). \quad (3.26)$$

The potential also has the a minimum $-H^2/2$ at $z = 0$ and a maximum $H^2/4$ at $z = \pm\infty$ (see Fig. 4). Eq. (3.25) with this potential can be turned into the following Schrödinger equation with a modified Pöschl-Teller potential:

$$\left[-\partial_z^2 - \frac{3H^2}{4} \text{sech}^2(Hz) \right] \rho_n = E_n \rho_n, \quad (3.27)$$

where $E_n = m_n^2 - \frac{H^2}{4}$. The energy spectrum of bound states can be expressed as follows:

$$E_n = -H^2\left(\frac{1}{2} - n\right)^2 \quad (3.28)$$

or, in terms of the squared mass:

$$m_n^2 = n(1 - n)H^2. \quad (3.29)$$

Here n is an integer satisfying $0 \leq n < \frac{1}{2}$. So there is only one bound state (the ground state), i.e., the normalized zero mode

$$\rho_0(z) = \sqrt{\frac{H}{\pi}} \text{sech}^2(Hz) \quad (3.30)$$

with $m_0^2 = 0$. The shape of the zero mode is shown in Fig. 5. It is easy to see that there is a mass gap between the zero mode and the first excited mode from Fig. 6 since the continuous spectrum of massive vector KK modes starts at $m^2 = H^2/4$ and asymptotically turn into plane waves.

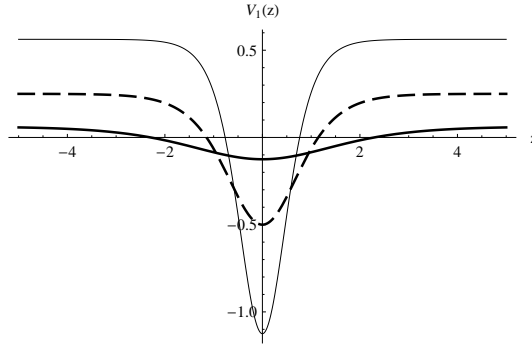


Figure 4: The shape of the potential of the vector fields $V_1(z)$. The parameter H is set to $H = 0.5$ for the thick line, $H = 1.0$ for the dashed line, and $H = 1.5$ for the thin line.

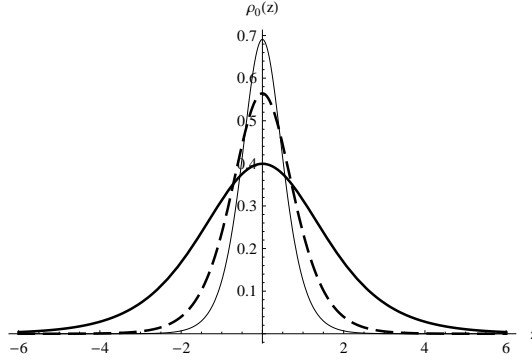


Figure 5: The shape of the vector zero mode $\rho_0(z)$. The parameter H is set to $H = 0.5$ for the thick line, $H = 1.0$ for the dashed line, and $H = 1.5$ for the thin line.

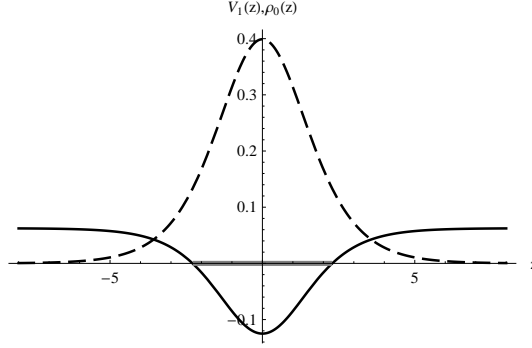


Figure 6: The shape of the potential of vector KK modes V_1 (thick line), the vector zero mode ρ_0 (dashed line), and the mass spectrum (thick grey lines) with the parameter $H = 0.5$.

3.3 Spin-1/2 fermion fields

Finally, we will study the localization of fermions on the pure de Sitter thick braneworld. In five-dimensional spacetime, fermions are four-component spinors and their Dirac structure can be described by $\Gamma^M = e^M_{\bar{M}} \Gamma^{\bar{M}}$ with $e^M_{\bar{M}}$ being the vielbein and $\{\Gamma^M, \Gamma^N\} = 2g^{MN}$. In this paper, $\bar{M}, \bar{N}, \dots = 0, 1, 2, 3, 5$ and $\bar{\mu}, \bar{\nu}, \dots = 0, 1, 2, 3$ denote the five-dimensional and four-dimensional local Lorentz indices, respectively, and $\Gamma^{\bar{M}}$ are the gamma matrices in five-dimensional flat spacetime. In our set-up, the vielbein is given by

$$e^{\bar{M}}_M = \begin{pmatrix} e^A \hat{e}^{\bar{\nu}}_{\mu} & 0 \\ 0 & e^A \end{pmatrix}, \quad (3.31)$$

$\Gamma^M = e^{-A}(\hat{e}^{\mu}_{\bar{\nu}} \gamma^{\bar{\nu}}, \gamma^5) = e^{-A}(\gamma^{\mu}, \gamma^5)$, where $\gamma^{\mu} = \hat{e}^{\mu}_{\bar{\nu}} \gamma^{\bar{\nu}}$, $\gamma^{\bar{\nu}}$ and γ^5 are the usual flat gamma matrices in the four-dimensional Dirac representation. The Dirac action of a spin-1/2 fermion with a mass term can be expressed as [64]

$$S_{\frac{1}{2}} = \int d^5x \sqrt{-g} [\bar{\Psi} \Gamma^M (\partial_M + \omega_M) \Psi - MF(z) \bar{\Psi} \Psi]. \quad (3.32)$$

Here ω_M is the spin connection defined as $\omega_M = \frac{1}{4} \omega^{\bar{M}\bar{N}}_M \Gamma_{\bar{M}} \Gamma_{\bar{N}}$ with

$$\begin{aligned} \omega^{\bar{M}\bar{N}}_M &= \frac{1}{2} e^{N\bar{M}} (\partial_M e^{\bar{N}}_N - \partial_N e^{\bar{M}}_M) - \frac{1}{2} e^{N\bar{N}} (\partial_M e^{\bar{M}}_N - \partial_N e^{\bar{M}}_M) \\ &\quad - \frac{1}{2} e^{P\bar{M}} e^{Q\bar{N}} (\partial_P e_{Q\bar{R}} - \partial_Q e_{P\bar{R}}) e^{\bar{R}}_M, \end{aligned} \quad (3.33)$$

and $F(z)$ is some general scalar function of the extra dimensional coordinate z . We will discuss about the properties of the scalar function $F(z)$ later on, in the context of the localization of KK fermion modes. The non-vanishing components of the spin connection ω_M for the background metric (3.1) are

$$\omega_{\mu} = \frac{1}{2} (\partial_z A) \gamma_{\mu} \gamma^5 + \hat{\omega}_{\mu}, \quad (3.34)$$

where $\hat{\omega}_{\mu} = \frac{1}{4} \hat{\omega}^{\bar{\mu}\bar{\nu}}_{\mu} \Gamma_{\bar{\mu}} \Gamma_{\bar{\nu}}$ is the spin connection derived from the metric $\hat{g}_{\bar{\mu}\bar{\nu}}(x) = \hat{e}^{\bar{\mu}}_{\mu}(x) \hat{e}^{\bar{\nu}}_{\nu}(x) \eta_{\bar{\mu}\bar{\nu}}$. Thus, the equation of motion corresponding to the action (3.32) can be written as

$$[\gamma^{\mu} (\partial_{\mu} + \hat{\omega}_{\mu}) + \gamma^5 (\partial_z + 2\partial_z A) - e^A MF(z)] \Psi = 0, \quad (3.35)$$

where $\gamma^\mu(\partial_\mu + \hat{\omega}_\mu)$ is the Dirac operator on the brane.

Next, we will investigate the five-dimensional Dirac equation (3.35), and write the spinor in terms of four-dimensional effective fields. On account of the fifth gamma matrix γ^5 , we anticipate the left- and right-handed projections of the four-dimensional part to behave differently. From Eq. (3.35), the solutions of the general chiral decomposition is found to be

$$\Psi = e^{-2A} \left(\sum_n \psi_{Ln}(x) L_n(z) + \sum_n \psi_{Rn}(x) R_n(z) \right), \quad (3.36)$$

where $\psi_{Ln}(x) = -\gamma^5 \psi_{Ln}(x)$ and $\psi_{Rn}(x) = \gamma^5 \psi_{Rn}(x)$ are the left-handed and right-handed components of a four-dimensional Dirac field, respectively. Hence, we assume that $\psi_{Ln}(x)$ and $\psi_{Rn}(x)$ satisfy the four-dimensional Dirac equations. Then the KK modes $L_n(z)$ and $R_n(z)$ should satisfy the following coupled equations:

$$[\partial_z + e^A M F(z)] L_n(z) = m_n R_n(z), \quad (3.37a)$$

$$[\partial_z - e^A M F(z)] R_n(z) = -m_n L_n(z). \quad (3.37b)$$

From the above coupled equations, we can obtain the Schrödinger-like equations for the left- and right-chiral KK modes of fermions:

$$(-\partial_z^2 + V_L(z)) L_n = m_{L_n}^2 L_n, \quad (3.38)$$

$$(-\partial_z^2 + V_R(z)) R_n = m_{R_n}^2 R_n, \quad (3.39)$$

where the mass-independent potentials are given by

$$V_L(z) = e^{2A} M^2 F^2(z) - e^A A' M F(z) - e^A M \partial_z F(z), \quad (3.40a)$$

$$V_R(z) = e^{2A} M^2 F^2(z) + e^A A' M F(z) + e^A M \partial_z F(z). \quad (3.40b)$$

For the purpose of getting the standard four-dimensional action for a massless fermion and a series of massive chiral fermions

$$\begin{aligned} S_{\frac{1}{2}} &= \int d^5x \sqrt{-g} \bar{\Psi} [\Gamma^M (\partial_M + \omega_M) - M F(z)] \Psi \\ &= \sum_n \int d^4x \sqrt{-\hat{g}} \bar{\psi}_n [\gamma^\mu (\partial_\mu + \hat{\omega}_\mu) - m_n] \psi_n, \end{aligned} \quad (3.41)$$

the following orthonormalization conditions for L_n and R_n are needed:

$$\int_{-\infty}^{+\infty} L_m L_n dz = \delta_{mn}, \quad (3.42)$$

$$\int_{-\infty}^{+\infty} R_m R_n dz = \delta_{mn}, \quad (3.43)$$

$$\int_{-\infty}^{+\infty} L_m R_n dz = 0. \quad (3.44)$$

If in the formulae (3.37a) and (3.37b) one sets $m_n = 0$, then one gets

$$L_0 \propto e^{-M \int e^A F dz}, \quad (3.45a)$$

$$R_0 \propto e^{M \int e^A F dz}. \quad (3.45b)$$

The above relations tell us that it is not possible to have both massless left- and right-chiral KK fermion modes localized on the brane at the same time, since when one is normalizable, the other one does not exist.

From Eqs. (3.38), (3.39) and (3.40), we can see that, if we do not introduce the mass term in the action (3.32), i.e., when $M = 0$, the potentials for left- and right-chiral KK modes $V_{L,R}(z)$ will vanish and both left- and right-chiral fermions cannot be localized on the thick brane. Moreover, if we demand $V_L(z)$ and $V_R(z)$ to be Z_2 -even with respect to the extra dimension z , then the mass term $MF(z)$ must be an odd function of z . In this paper, we will consider four cases: $F(z) = \varepsilon(z)$, $F(z) = \tanh(Hz)$, $F(z) = \sinh(Hz)$ and $F(z) = (Hz)^{2k+1}e^{-A}$.

3.3.1 Case I: $F(z) = \varepsilon(z)$

Firstly, we will consider the simplest case $F(z) = \varepsilon(z)$ [64], where $\varepsilon(z \neq 0) \equiv \frac{z}{|z|}$ and $\varepsilon(0) = 0$. The explicit forms of the potentials (3.40) can be expressed as follows:

$$V_L(z) = \frac{H^2 M \text{sech}^2(Hz)}{b^2} [M + b \sinh(Hz) \varepsilon(z)] - 2\delta(z), \quad (3.46a)$$

$$V_R(z) = \frac{H^2 M \text{sech}^2(Hz)}{b^2} [M - b \sinh(Hz) \varepsilon(z)] + 2\delta(z). \quad (3.46b)$$

From these expressions, we find that the potentials $V_{L,R}(z)$ have the asymptotic behavior: $V_{L,R}(z \rightarrow \pm\infty) \rightarrow 0$. Because there is a δ function in the expressions for both potentials, when $z = 0$, $V_L(0) = -\infty$ and $V_R(0) = +\infty$. It is clear that only the potential $V_L(z)$ is of volcano type and has a δ -potential well at the location of the brane, so just the massless mode of left-chiral fermion might be trapped on the brane. The left-chiral fermion zero mode can be computed by solving (3.37a) with $m_0 = 0$:

$$\begin{aligned} L_0(z) &\propto \exp\left(-\int_0^z dz' e^{A(z')} M \varepsilon(z')\right) \\ &= \exp\left(-\frac{2M}{b} \arctan\left[\tanh\left(\frac{H|z|}{2}\right)\right]\right). \end{aligned} \quad (3.47)$$

However, from this expression, we know that when far away from the brane, $L_0(z \rightarrow \pm\infty)$ approaches a positive constant $e^{-\frac{\pi M}{2b}}$, a fact which results in the non-normalization of $L_0(z)$. So the zero mode of left-chiral fermion cannot be localized on the brane. The shape of the zero mode $L_0(z)$ is shown in Fig. 7. Hence, both left- and right-chiral fermion zero modes cannot be trapped on the brane. Thus, there is no mass gap in the spectrum of KK modes for both left- and right-chiral fermions, the spectrum is continuous and starts at $m^2 = 0$.

3.3.2 Case II: $F(z) = \tanh(Hz)$

For the case $F(z) = \tanh(Hz)$, the explicit forms of the potentials (3.40) are

$$V_L(z) = \frac{H^2 M}{4b^2} \text{sech}^4(Hz) [4M \sinh^2(Hz) + b \cosh(3Hz) - 5b \cosh(Hz)], \quad (3.48)$$

$$V_R(z) = \frac{H^2 M}{4b^2} \text{sech}^4(Hz) [4M \sinh^2(Hz) - b \cosh(3Hz) + 5b \cosh(Hz)]. \quad (3.49)$$

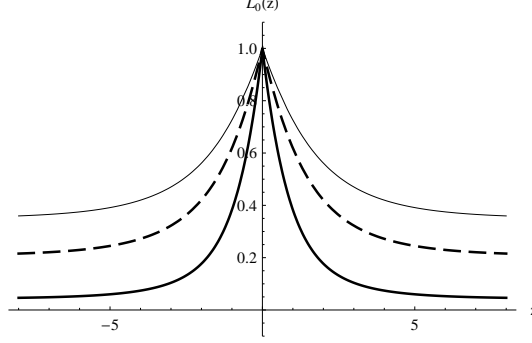


Figure 7: The shape of the zero mode of left-chiral fermions $\rho_0(z)$ for different values of the parameter b . The parameters are set to $M = 1.0$, $H = 0.5$, and $b = 0.5$ for the thick line, $b = 1.0$ for the dashed line, and $b = 1.5$ for the thin line.

It can be seen that the potentials $V_{L,R}$ have the following asymptotic behaviors: they tend to zero as $z \rightarrow \pm\infty$, and at $z = 0$, the potential V_L reaches its minimum (negative value) $-H^2M/b$ while the potential V_R has its maximum (positive value) H^2M/b (see Fig. 8). So $V_L(z)$ is a modified volcano-type potential. For this type of potentials, there is no mass gap to separate the fermion zero mode from the excited KK massive modes. Both left- and right-chiral KK modes have a continuous gapless spectrum. Because only the potential for left-chiral fermions has a negative value at the location of the brane, we only need to study whether the zero mode of left-chiral fermions L_0 could be localized on the brane. The expression for L_0 is

$$L_0 \propto \exp \left[\frac{M}{b} (\text{sech}(Hz) - 1) \right]. \quad (3.50)$$

From this expression and Fig. 9, we can see that the zero mode $L_0 \rightarrow e^{-\frac{M}{b}} > 0$ as $z \rightarrow \pm\infty$, which indicates that the normalization condition $\int_{-\infty}^{+\infty} L_0^2(z) dz < \infty$ is not satisfied, so the zero mode of the left-chiral fermions cannot be localized on the brane.

3.3.3 Case III: $F(z) = \sinh(Hz)$

For the choice $F(z) = \sinh(Hz)$, the potentials (3.40) can be expressed as follows:

$$V_L(z) = \frac{H^2M}{b^2} [M - (b + M)\text{sech}^2(Hz)], \quad (3.51)$$

$$V_R(z) = \frac{H^2M}{b^2} [M + (b - M)\text{sech}^2(Hz)]. \quad (3.52)$$

From Fig. 10, we can see that both potentials have the same asymptotic behavior: $V_{L,R} \rightarrow \frac{H^2M^2}{b^2}$ when $z \rightarrow \pm\infty$. At $z = 0$, the potential of left-chiral fermions has a minimum with negative value $-\frac{H^2M}{b}$, however, the potential of right-chiral fermions has a positive value one: $\frac{H^2M}{b}$ if and only if $M > b$. Thus, just the potential of left-chiral fermions possesses a negative value at the location of the brane, so only the zero mode of left-chiral fermions

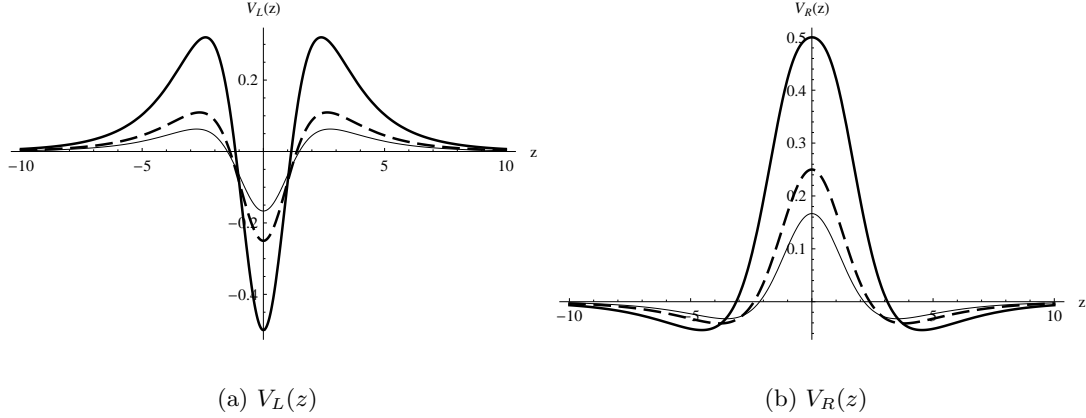


Figure 8: The shape of the potential for the left-chiral fermions $V_L(z)$ is displayed in (a), and for the right-chiral fermions $V_R(z)$ is plotted in (b) for the case II. The parameters are set to $H = 0.5$, $M = 1$, $b = 0.5$ for the thick lines, $b = 1.0$ for the dashed lines and $b = 1.5$ for the thin lines.

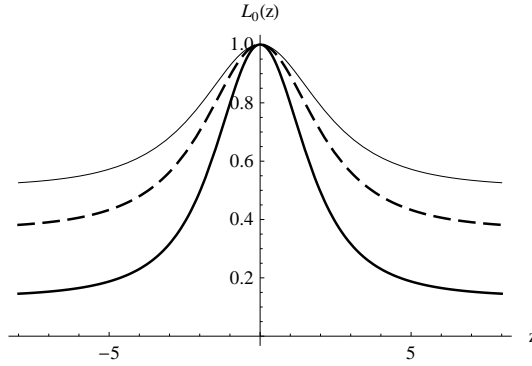


Figure 9: The shape of the massless left-chiral fermion $L_0(z)$ for case II. The parameters are set to $H = 0.5$, $M = 1$, $b = 0.5$ for the thick line, $b = 1.0$ for the dashed line and $b = 1.5$ for the thin line.

L_0 will be localized on the brane. The solution for L_0 can be written as

$$L_0(z) = \left[\frac{H \Gamma\left(\frac{b+M}{2b}\right)}{\sqrt{\pi} \Gamma\left(\frac{M}{2b}\right)} \right]^{\frac{1}{2}} \text{sech}^{\frac{M}{b}}(Hz). \quad (3.53)$$

Because the parameter M and b are positive, the zero mode L_0 vanishes at $z \rightarrow \pm\infty$, and it satisfies the normalization condition. So the massless mode of left-chiral fermions is localized on the brane. We plot the shape of the left-chiral fermion zero mode in Fig. 11. However, the zero mode of right-chiral fermions does not exist, a fact which can be seen from the potential (3.52) and Fig. 10(b).

We can see that for $m_{L_n}^2 > \frac{H^2 M^2}{b^2}$, the left-chiral fermions can be described by asymptotic plane waves. The general bound KK modes L_n for the potential (3.51) can be found to be

$$L_n \propto \cosh^{1+\frac{M}{b}}(Hz) {}_2F_1\left(a_n, b_n; \frac{1}{2}; -\sinh^2(Hz)\right) \quad (3.54)$$

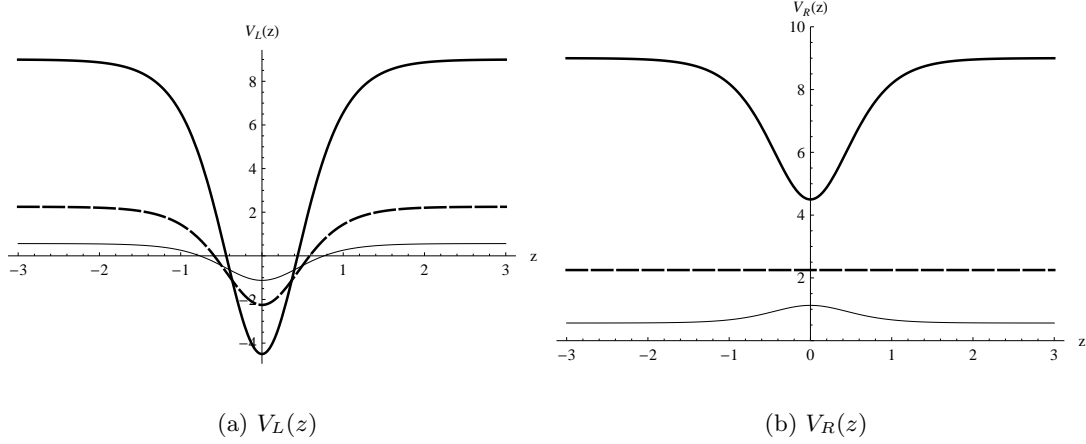


Figure 10: The shape of the potential $V_L(z)$ is presented in (a) and $V_R(z)$ is displayed in (b) for the case III. The parameters are set to $H = 1.5$, $M = 1$, $b = 0.5$ for thick lines, $b = 1.0$ for dashed lines and $b = 1.5$ for thin lines.

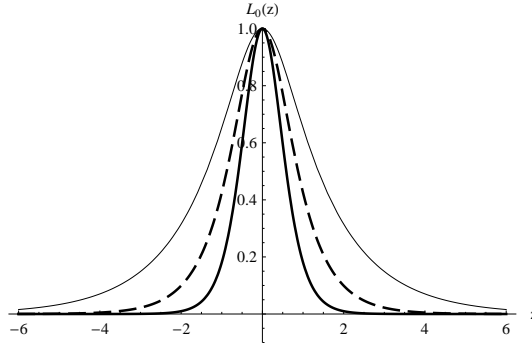


Figure 11: The shape of the zero mode of the left-chiral fermion $L_0(z)$ for the case III. The parameters are set to $H = 1.5$, $M = 1$, $b = 0.5$ for the thick line, $b = 1.0$ for the dashed line and $b = 1.5$ for the thin line.

for even n and

$$L_n \propto \cosh^{1+\frac{M}{b}}(Hz) \sinh(Hz) {}_2F_1\left(a_n + \frac{1}{2}, b_n + \frac{1}{2}; \frac{3}{2}; -\sinh^2(Hz)\right) \quad (3.55)$$

for odd n , where ${}_2F_1$ is the hypergeometric function, and the parameters a_n and b_n are given by

$$a_n = \frac{1}{2}(n+1), \quad b_n = \frac{M}{b} - \frac{1}{2}(n-1). \quad (3.56)$$

The corresponding mass spectrum of the bound states is

$$m_{L_n}^2 = H^2 \left(\frac{2M}{b} - n \right) n, \quad \left(n = 0, 1, 2, \dots < \frac{M}{b} \right). \quad (3.57)$$

It can be seen that the ground state always belongs to the spectrum of left-chiral KK modes, which is precisely the zero mode (3.53) with $m_{L_0}^2 = 0$. Because the ground state

has the lowest squared mass $m_{L_0}^2 = 0$, there are no tachyonic left-chiral fermions. If $M < b$, there is only one bound state, i.e., the zero mode (3.53). In order to get massive excited bound states, the condition $M > b$ should be satisfied.

For the potential of right-chiral fermions $V_R(z)$ (3.52), from Fig. 10(b), we can see that $V_R(z)$ is always positive near the location of the brane, which shows that it cannot trap the zero mode of right-chiral fermions. For the case $M < b$, the potential V_R has a maximum $\frac{H^2 M}{b}$ at $z = 0$, and has a minimum $\frac{H^2 M^2}{b^2}$ at $z \rightarrow \pm\infty$, i.e., $0 < V_R(z \rightarrow \pm\infty) < V_R(z = 0)$. So in this case, there is no any bound state for right-chiral fermions. For the special case $M = b$, the potential V_R is a positive constant: $V_R(z) = \frac{H^2 M^2}{b^2}$. Hence, there is still no any bound state for this case. In the last case $M > b$, we can see that $0 < V_R(z = 0) < V_R(z \rightarrow \pm\infty)$, which indicates that there exist some bound states, but the ground state is a massive one:

$$R_0 = \left[\frac{H \Gamma\left(\frac{M}{2b}\right)}{\sqrt{\pi} \Gamma\left(\frac{M-b}{2b}\right)} \right]^{\frac{1}{2}} \cosh^{1-\frac{M}{b}}(Hz), \quad (M > b) \quad (3.58)$$

with the mass determined by $m_{R_0}^2 = H^2 \left(\frac{2M}{b} - 1\right) > H^2 > 0$. The general bound states for this case ($M > b$) are

$$R_n(z) \propto \cosh^{\frac{M}{b}}(Hz) {}_2F_1\left(\frac{1+n}{2}, \frac{M}{b} - \frac{1+n}{2}; \frac{1}{2}; -\sinh^2(Hz)\right) \quad (3.59)$$

for even n and

$$R_n(z) \propto \cosh^{\frac{M}{b}}(Hz) \sinh(Hz) {}_2F_1\left(1 + \frac{n}{2}, \frac{M}{b} - \frac{n}{2}; \frac{3}{2}; -\sinh^2(Hz)\right) \quad (3.60)$$

for odd n . Then, the corresponding mass spectrum is

$$m_{R_n}^2 = H^2 \left(\frac{2M}{b} - (n+1)\right) (n+1), \quad \left(M > b, \quad n = 0, 1, 2, \dots < \frac{M}{b} - 1\right). \quad (3.61)$$

By comparing to the mass spectrum of the left-chiral fermions (3.57), we come to the following conclusion: the number of bound KK modes of right-chiral fermions N_R is one less than that of the left ones N_L , i.e. $N_R = N_L - 1$.

When $M < b$, there is only one left-chiral fermion bound KK mode (the zero mode), so only the four-dimensional massless left-chiral fermion can be localized on the brane. When $M > b$, there are N ($N = N_L \geq 2$) bound left-chiral fermion KK modes and $N - 1$ ($N_R = N - 1$) right-chiral ones. Hence, we obtain that the four-dimensional massless left-chiral fermion and the massive Dirac fermions consisting of pairs of coupled left- and right-chiral KK modes can be localized on the brane. The KK modes L_n and R_n are plotted in Fig. 12 and Fig. 13, respectively, and the corresponding mass spectra are shown in Fig. 14.

3.3.4 Case IV: $F(z) = (Hz)^{2k+1} e^{-A(z)}$

Now let us consider the following family of functions:

$$F(z) = (Hz)^{2k+1} e^{-A(z)}, \quad (3.62)$$

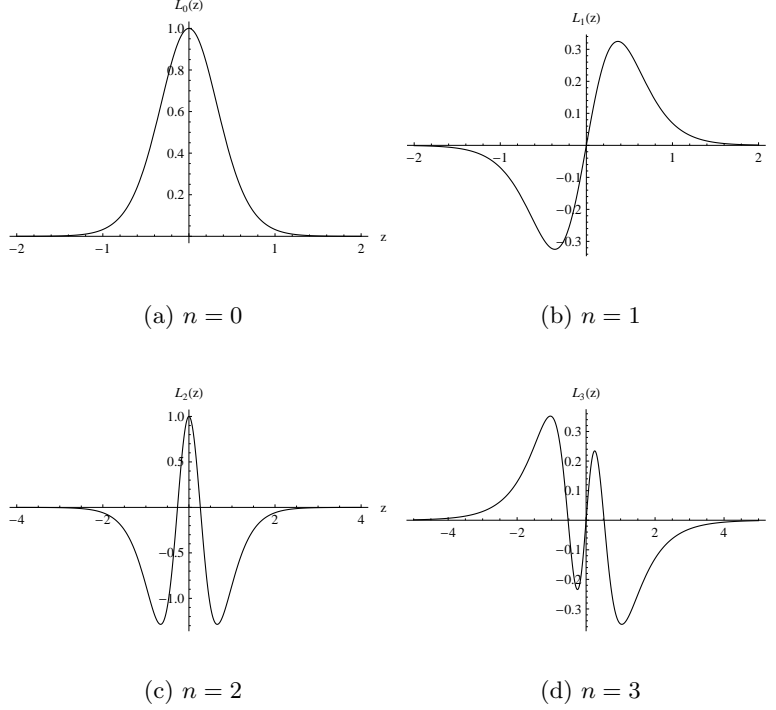


Figure 12: The shape of the left-chiral KK modes L_n for $0 \leq n \leq 3$ in case III. The parameters are set to $M = 1.0$, $b = 0.25$ and $H = 1.5$.

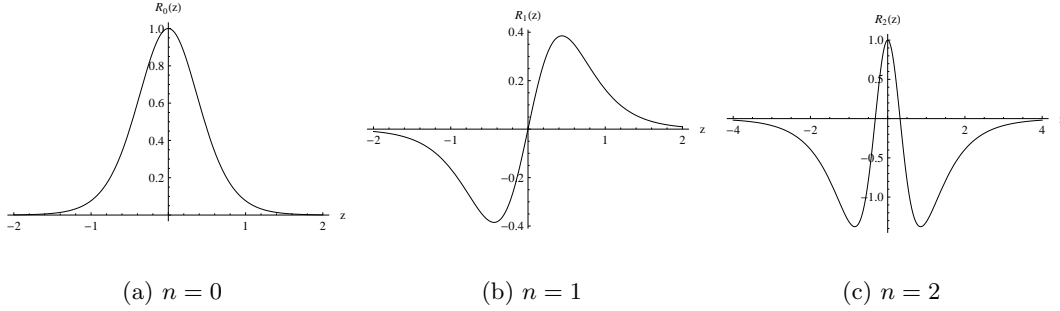


Figure 13: The shape of the right-chiral KK modes R_n for $0 \leq n \leq 2$ in case III. The parameters are set to $M = 1.0$, $b = 0.25$ and $H = 1.5$.

where the factor $H z$ renders a dimensionless function $F(z)$ and $k = 0, 1, 2, \dots$.

By using (3.4), the explicit form of the potentials (3.40) can be expressed as follows

$$V_L(z) = MH \left[MH(Hz)^{4k} z^2 - (2k+1)(Hz)^{2k} \right], \quad (3.63a)$$

$$V_R(z) = MH \left[MH(Hz)^{4k} z^2 + (2k+1)(Hz)^{2k} \right]. \quad (3.63b)$$

Both potentials have the same asymptotic behavior $V_{L,R} \rightarrow \infty$ when $z \rightarrow \pm\infty$ and are bounded below as is shown in Figs. 15 and 16 for different values of $k > 0$, the case $k = 0$ will be considered below as a separate one. Then, the KK spectrum is discrete for both

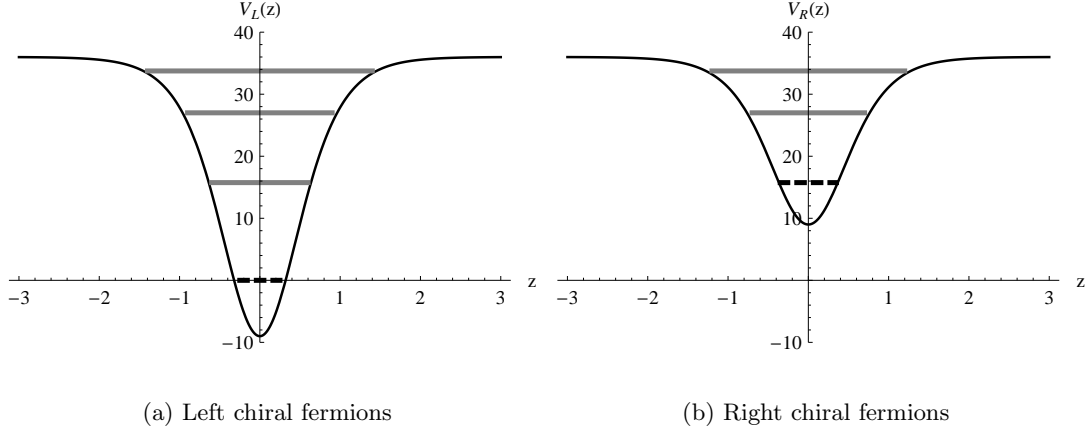


Figure 14: The shape of the potentials $V_{L,R}$ (thick lines) and the mass spectra (the dashed lines correspond to $m_{L_0}^2 = 0$ and $m_{R_0}^2 = 0$, and the grey lines correspond to the bound massive levels) for the case III. The parameters are set to $M = 1.0$, $b = 0.25$ and $H = 1.5$.

kind of fermions. It is possible to show using (3.45a) that the zero left-chiral KK mode has the following form

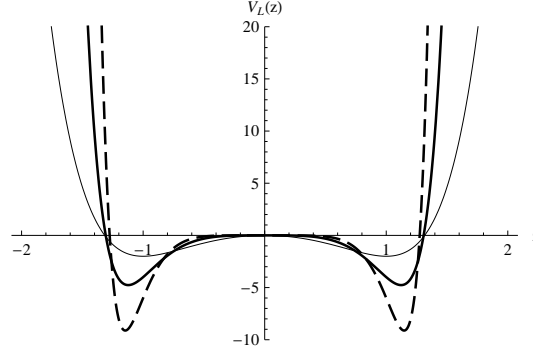


Figure 15: The shape of the potential V_L is represented for different values of k : $k = 1$ (thin line), $k = 2$ (thick line) and $k = 3$ (dashed line). In all these cases the potential V_L has two negative minima and becomes infinite as $z \rightarrow \pm\infty$, resembling a shifted mexican hat potential; thus, there is a localized massless KK mode and a tower of discrete massive modes. The parameters are set to $H = 1$ and $M = 1$ in the figure.

$$L_0 \propto e^{-\frac{M}{2H(k+1)}(Hz)^{2(k+1)}}. \quad (3.64)$$

Thus, the massless left-chiral KK mode is localized on the brane (see Fig. 17), while the massless right-chiral fermion does not exist. The class of functions (3.62) describe a model where both type of fermions are localized on the brane and only the left-chiral KK spectrum possesses a massless mode.

Let us consider the simplest case $k = 0$ in (3.62), the potentials (3.63a) and (3.63b)

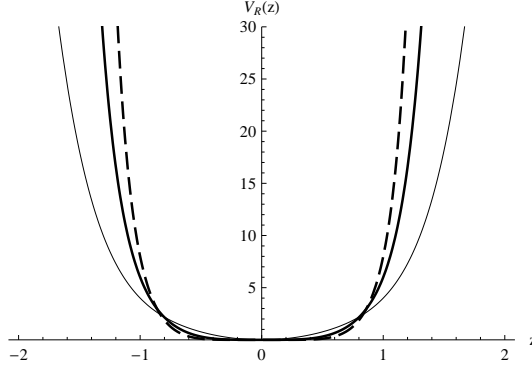


Figure 16: The shape of the potential V_R is represented for different values of k : $k = 1$ (thin line), $k = 2$ (thick line) and $k = 3$ (dashed line). In all cases there is a single minimum of the potential V_R at $z = 0$ and it is infinite as $z \rightarrow \pm\infty$. In this case we do not have a localized massless zero mode and all massive KK modes are localized on the brane. Here, we also set $H = 1$ and $M = 1$.

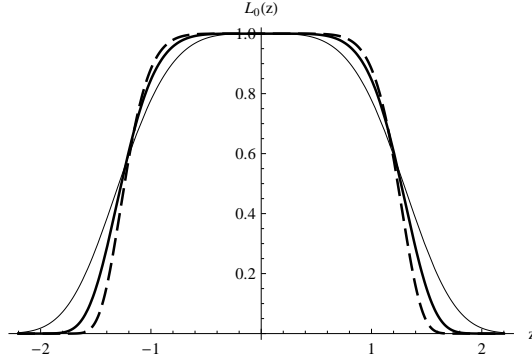


Figure 17: The shape of the massless left-chiral KK mode $L_0(z)$ is represented for different values of k : $k = 1$ (thin line), $k = 2$ (thick line) and $k = 3$ (dashed line). The parameters are set to $H = 1$ and $M = 1$.

take the form:

$$V_L(z) = H^2 M^2 z^2 - HM, \quad (3.65a)$$

$$V_R(z) = H^2 M^2 z^2 + HM. \quad (3.65b)$$

Both potentials constitute shifted one-dimensional quantum harmonic oscillator potentials and are plotted in Figs. 18 and 19, respectively. The corresponding Schrödinger-like equations for the potentials $V_{L,R}$ can be written as:

$$-\partial_z^2 L_n(z) + H^2 M^2 z^2 L_n(z) = (m_{L_n}^2 + HM) L_n(z), \quad (3.66a)$$

$$-\partial_z^2 R_n(z) + H^2 M^2 z^2 R_n(z) = (m_{R_n}^2 - HM) R_n(z). \quad (3.66b)$$

Thus, the above equations describe a quantum harmonic oscillator in one dimension if we

define the effective energy modes $E_{L,R}$ and the effective potentials $\bar{V}_{L,R}$ as follows

$$E_{L_n} = \frac{1}{2} \left(\frac{m_{L_n}^2}{M} + H \right), \quad (3.67a)$$

$$E_{R_n} = \frac{1}{2} \left(\frac{m_{R_n}^2}{M} - H \right), \quad (3.67b)$$

$$\bar{V}_L(z) = \bar{V}_R(z) = H^2 M^2 z^2. \quad (3.67c)$$

We see that both potentials $\bar{V}_{L,R}$ are equal, then, they have the same asymptotic behavior $\bar{V}_{L,R} \rightarrow \pm\infty$ when $z \rightarrow \pm\infty$ and have their minima at $z = 0$; thus, the potentials $\bar{V}_{L,R}$ admit the same number of bound energy modes for the left- and right-chiral fermions. The corresponding spectra of energies for the equations (3.66) are given by

$$E_{L_n} = H \left(n + \frac{1}{2} \right) = \frac{1}{2} \left(\frac{m_{L_n}^2}{M} + H \right), \quad (3.68a)$$

$$E_{R_n} = H \left(n + \frac{1}{2} \right) = \frac{1}{2} \left(\frac{m_{R_n}^2}{M} - H \right), \quad (3.68b)$$

where $n = 0, 1, 2, \dots$. Then, the spectra for the left and right masses $m_{L,R}^2$ are $m_{L_n}^2 = 2MHn$ and $m_{R_n}^2 = 2MH(n+1)$, respectively, as indicated in Figs. 18 and 19.

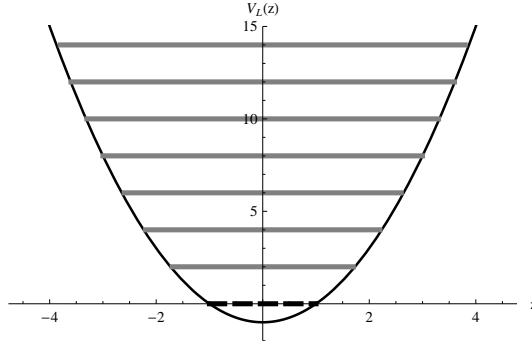


Figure 18: The shape of the potential V_L is represented for $k = 0$; the dashed line corresponds to $m_{L_0}^2 = 0$ and the first seven massive levels ($1 \leq n \leq 7$) of the $m_{L_n}^2$ spectrum are given by the grey lines. The parameters are set to $H = 1$ and $M = 1$ in the figure.

Like in the case III, we see that the KK mass spectrum for the right-chiral fermions does not include a massless mode, while the left-chiral fermions contains the massless KK mode. By rewriting equations (3.66) we see that both of them represent quantum harmonic oscillators along the fifth dimension

$$-\partial_z^2 L_n(z) + H^2 M^2 z^2 L_n(z) = E_{L_n} L_n(z), \quad (3.69a)$$

$$-\partial_z^2 R_n(z) + H^2 M^2 z^2 R_n(z) = E_{R_n} R_n(z). \quad (3.69b)$$

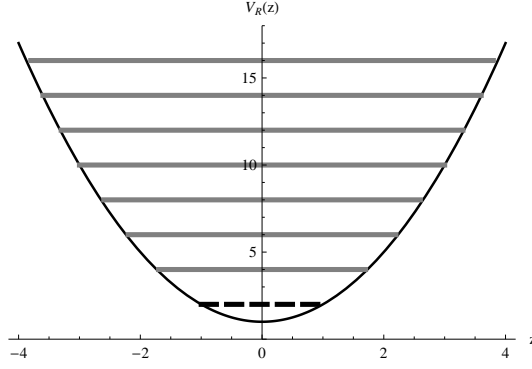


Figure 19: Here the shape of the potential V_R is represented for $k = 0$; the dashed line corresponds to the massive state $m_{R_0}^2 = 2MH$ and the first seven massive levels ($1 \leq n \leq 7$) of the $m_{R_n}^2$ spectrum are given by the grey lines. The parameters are set to $H = 1$ and $M = 1$ in the figure.

The zero mode can be localized for both the left- and right-chiral fermions and has the explicit form

$$L_0(z) = \frac{(HM)^{\frac{1}{2}}}{\sqrt{2\pi}} e^{-\frac{1}{2}HMz^2}, \quad (3.70a)$$

$$R_0(z) = \frac{(HM)^{\frac{1}{2}}}{\sqrt{2\pi}} e^{-\frac{1}{2}HMz^2}, \quad (3.70b)$$

where (3.70a) is a massless bound state, while (3.70b) is a massive one.

The total spectra for the equations (3.69) is expressed in terms of Hermite polynomials by the next eigenfunctions

$$L_n(z) = \frac{B_L}{2^n} (HM)^{\frac{1}{4}} H_n(\sqrt{HM}z) e^{-\frac{1}{2}HMz^2}, \quad (3.71a)$$

$$R_n(z) = \frac{B_R}{2^n} (HM)^{\frac{1}{4}} H_n(\sqrt{HM}z) e^{-\frac{1}{2}HMz^2}, \quad (3.71b)$$

where B_L and B_R are normalization constants.

The spectrum for both type of fermions is discrete and the separation between two contiguous KK modes is given by

$$\Delta m_n = \frac{\sqrt{2HM}}{\sqrt{n+1} + \sqrt{n}}. \quad (3.72)$$

The above relations tell us that if $n \rightarrow \infty$, $\Delta m_n \rightarrow 0$, then, for higher KK massive modes the spectrum is quasi-continuous.

The previous result shows that by choosing $F(z) = Hze^{-A(z)}$, it is possible to localize on the brane the whole discrete KK spectrum: both massless and massive modes for the left-chiral fermions and just the massive ones for the right-chiral ones.

4. Conclusion and discussion

In this paper, the localization and mass spectra of various bulk matter fields on a thick brane generated by pure constant curvature in 4D and 5D, without the inclusion of scalar

fields, has been investigated. For scalar and vector fields, both the scalar and vector zero modes can be localized on the thick brane, and there exists a mass gap in the respective spectra. For scalar fields, the spectrum consists of a massless mode (the ground state), a bound excited KK mode and a series of continuous massive KK modes. For vector fields, the spectrum consists of a massless mode (the ground state) and a series of continuous massive KK modes. For the localization of a fermion zero mode, we must introduce the mass term $MF(z)\Psi\bar{\Psi}$ in the five-dimensional action. Four cases have been investigated: For $F(z) = \varepsilon(z)$ and $F(z) = \tanh(Hz)$, the fermion zero mode cannot be localized on the brane. For $F(z) = \sinh(Hz)$, only the left-chiral fermion zero mode can be localized on the brane and there exist a mass gap for both left- and right-chiral fermions. The *finite* number of bound massive KK modes of left- and right-chiral fermions is the same, and it is determined by the ratio M/b . Hence, the massless fermion localized on the brane consists of just the left-chiral KK mode, while the massive fermions localized on the brane consist of left- and right-chiral KK modes and constitute the four-dimensional Dirac massive fermions. For the case in which $F(z) = (Hz)^{2k+1}e^{-A(z)}$ we qualitatively show that all the mass spectra for the left- and right-chiral KK modes are *infinite*, discrete and are localized on the brane. For $k > 0$ it is difficult to solve the eigenvalue problem for KK modes exactly, thus, we consider the simplest case where $k = 0$ and we found that it resembles the one-dimensional quantum harmonic oscillator problem in which the squared mass gap $\Delta m_{L,R}^2$ between two contiguous states for left- and right-chiral fermions is equidistant. As we can see from the expressions for the $m_{L_n}^2$ and $m_{R_n}^2$, all the right-chiral fermion spectrum is shifted in $2MH$ with respect to the left-chiral fermion spectrum, while, as it is shown in (3.72), the mass gap tends to vanish $\Delta m_n \rightarrow 0$ when $n \rightarrow \infty$.

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